Intro. to ODEs

Quiz 3 **Solutions**

1) Consider (but do not try to solve) the following initial value problem.

$$\frac{dy}{dx} = \frac{\sqrt[3]{4-y^2}}{\ln(x)}$$
$$y(x_0) = y_0$$

a) What restrictions (if any) need to be placed on $x_0$ and $y_0$ so that the existence of a solution is guaranteed?

In order for the right-hand side of the ODE to be continuous (and real-valued) near the initial datum, we must have $x_0 > 0$ and $x_0 \neq 1$.

b) What quantity do we need to compute to examine uniqueness of solutions? Compute that quantity.

$$\frac{\partial f}{\partial y} = -\frac{2y}{3\ln(x)(4-y^2)^{2/3}}$$

c) What restrictions (if any) need to be placed on $x_0$ and $y_0$ so that the solution is guaranteed to be unique.

In addition to $x_0 > 0$ and $x_0 \neq 1$, we cannot be sure of uniqueness if $y_0 = \pm 2$.

2) Find the general solution to the following first order differential equation.

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

Since this is a first-order homogeneous equation, we make the substitution $y = xv$ which gives $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

$$v + x\frac{dv}{dx} = \frac{x^2 v}{x^2 + x^2 v^2} = \frac{v}{1+v^2}$$
$$x\frac{dv}{dx} = -\frac{v^3}{1+v^2}$$
$$\int \frac{1+v^2}{v^3}\,dv = -\int \frac{dx}{x}$$
$$-\frac{1}{2}v^{-2} + \ln|v| = -\ln|x| + C$$
$$-\frac{1}{2}\left(\frac{x}{y}\right)^2 + \ln\left|\frac{y}{x}\right| = -\ln|x| + C$$
$$-\frac{1}{2}\left(\frac{x}{y}\right)^2 + \ln|y| = C$$

3) Solve the following initial value problem.

$$\frac{dy}{dx} + xy = xy^4$$
$$y(0) = \frac{1}{2}$$

This is a Bernoulli equation, and so we make the substitution $z = y^{1-4} = y^{-3}$. This means $y = z^{-1/3}$ and

$$\frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{3}y^4\frac{dz}{dx}.$$

This transforms the ODE into

$$-\frac{1}{3}y^4\frac{dz}{dx} + xy = xy^4$$
$$\frac{dz}{dx} - 3xz = -3x$$

which is a first-order ODE for $z$. Since the integrating factor is

$$\mu(x) = e^{-\int 3x\,dx} = e^{-3x^2/2},$$

the solution for $z$ is

$$z(x) = \frac{\int -3xe^{-3x^2/2}\,dx + C}{e^{-3x^2/2}} = \frac{e^{-3x^2/2} + C}{e^{-3x^2/2}} = Ce^{3x^2/2} + 1$$

This means the general solution of the original ODE is

$$y(x) = \left(Ce^{3x^2/2} + 1\right)^{-1/3}.$$

Fitting the initial condition gives

$$y(0) = (C+1)^{-1/3} = \frac{1}{2}$$
$$C + 1 = 2^3 = 8$$
$$C = 7.$$

So, the solution to our initial value problem is

$$y(x) = \left(7e^{3x^2/2} + 1\right)^{-1/3}.$$